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The Ward ansätze and Painlevé tau function

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Abstract

We have classified a tau function for the hypergeometric solutions of the Painlevé VI equation constructed by Shah and Woodhouse (2006 *J. Phys. A: Math. Gen.* **39**, 12265–9) through twistor methods. We have shown that the tau function is the product of a Toeplitz determinant and a power of the time variable t. In a suitable trivialization of the twistor bundle, the symbol of this Toeplitz determinant is the minus of the off-diagonal entry in the patching matrix. The method can also be applied to other solutions obtained from the Ward ansätze.

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1. The twistor reduction of the anti-self-dual Yang–Mills equation and the Painlevé VI equation

Many integrable systems can be obtained from reductions of other integrable systems and inherit the integrability from their mother systems. A large class of integrable systems can be represented as reductions of the anti-self-dual Yang–Mills (ASDYM) system via conformal vector fields [4]. In particular, the six Painlevé equations are equivalent to reductions of the ASDYM equation through the action of three conformal Killing vectors. This opens up a new way of studying the Painlevé equations through twistor theory [6, 15, 16]. In [6], special solutions of the Painlevé VI equation were constructed in a simple way through the use of the Ward ansätze in twistor theory. In this paper, we will classify the tau function of these solutions in terms of Toeplitz determinants. We will show that the tau function of a solution to the Painlevé VI constructed from a twistor bundle with a patching matrix of the form (2.1)

$$T_k(\lambda, \mu, \zeta) = \begin{pmatrix} \zeta^k & \phi \\ 0 & \zeta^{-k} \end{pmatrix}$$

is given by $f(t)D_k[-\phi]$, where $f(t) = C_1 t^{-k}$ if $|t| < 1$ and $f(t) = C_2$ if $|t| > 1$ for some constants C_1 and C_2, while $D_k[-\phi]$ is the k-dimensional Toeplitz determinant with the symbol $-\phi$. The result is stated in theorem 2. The discontinuity in the expression reflects the fact that the Toeplitz determinant has a singularity when $|t| = 1$.

Let us consider the complexified Minkowski space $\mathbb{CM}$ with complex coordinate $(w, \tilde{w}, z, \tilde{z})$ and metric

$$\mathrm{d}s^2 = 2\,(\mathrm{d}z\,\mathrm{d}\tilde{z} - \mathrm{d}w\,\mathrm{d}\tilde{w})\,.$$

A connection

$$\mathrm{d} + \Phi = \mathrm{d} + \Phi_w\,\mathrm{d}w + \Phi_{\tilde{w}}\,\mathrm{d}\tilde{w} + \Phi_z\,\mathrm{d}z + \Phi_{\tilde{z}}\,\mathrm{d}\tilde{z} \tag{1.1}$$

on $\mathbb{CM}$ is anti-self-dual if the connection is flat on self-dual null planes (α-planes) whose tangent spaces are spanned by the vector fields

$$l = \partial_w - \zeta\partial_{\tilde{z}}, \qquad m = \partial_z - \zeta\partial_{\tilde{w}}$$

for some complex number ζ. If, in addition, the connection is invariant under the action of the following conformal Killing vectors

$$X_1 = -z\partial_z - w\partial_w, \qquad X_2 = -\tilde{w}\partial_{\tilde{w}} - \tilde{z}\partial_{\tilde{z}}, \qquad X_3 = z\partial_z + \tilde{w}\partial_{\tilde{w}}, \tag{1.2}$$

the ASDYM equation is equivalent to the Painlevé VI equation

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t}\\ &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right),\end{aligned} \tag{1.3}$$

where α, β, γ and δ are some constants. In terms of the spacetime variables, $t = \frac{\tilde{z}z}{w\tilde{w}}$. The relation between y and the connection $\mathrm{d} + \Phi$ can be found in chapter 7 of [4].

In this case, the ASDYM condition is equivalent to isomonodromic deformations of the system of linear ODE [6]:

$$\frac{\mathrm{d}g}{\mathrm{d}\zeta} = \left(\frac{A + \tilde{B}}{\zeta + r} + \frac{\tilde{A} + B}{\zeta + s} - \frac{\tilde{A} + \tilde{B}}{\zeta}\right) g, \tag{1.4}$$

where $r = \frac{w}{\tilde{z}}$, $s = \frac{z}{\tilde{w}}$, A, $\tilde{A}$, B and $\tilde{B}$ are functions of t only. The Painlevé VI equation can be recovered by fixing the values of w, $\tilde{w}$ and $\tilde{z}$ to be $\tilde{w} = w = \tilde{z} = 1$ and let $z = t$. Different choices of the variables will result in gauge equivalent systems of ODE [4].

2. The Ward ansätze and hypergeometric solution of Painlevé VI

A classical result of Penrose and Ward [11] identifies the solutions of the ASDYM equation with holomorphic vector bundles in a neighborhood $\mathcal{P}$ of a line $\hat{x} \in \mathbb{CP}^3$ that is trivial on the line. Let λ, μ be $\lambda = w + \zeta\tilde{z}$ and $\mu = z + \zeta\tilde{w}$. The Ward ansätze [12] is an ansätze that uses a patching matrix of the vector bundle of the form

$$T_k(\lambda, \mu, \zeta) = \begin{pmatrix} \zeta^k & \phi \\ 0 & \zeta^{-k} \end{pmatrix} \tag{2.1}$$

to construct solutions of the ASDYM equation, where k is some non-negative integer and ϕ is a function of λ, μ and ζ only. The key is to make a Birkhoff factorization of the patching matrix

$$T(\lambda, \mu, \zeta) = H_\infty^{-1} H_0 \tag{2.2}$$

at each fixed spacetime point z, $\tilde{z}$, w and $\tilde{w}$, where H_∞ is holomorphic in a neighborhood V_∞ of $\zeta = \infty$ and H_0 is holomorphic in a neighborhood V_0 of $\zeta = 0$. The connection (1.1) can then be recovered by evaluating H_∞ at $\zeta = \infty$ and H_0 at $\zeta = 0$ [4, 10].

In [6], invariant solutions corresponding to the Ward ansätze were constructed from solutions of hypergeometric equations. We shall see that these solutions correspond to the solutions whose tau functions are given by Toeplitz determinants with symbol $-\phi$.

3. Toeplitz determinant and Ward ansätze

The factorization problem can be thought of as a Riemann–Hilbert problem and is related to the Toeplitz determinant with the symbol ϕ. Recall that a k-dimensional Toeplitz matrix with symbol ϕ is a matrix $T_k(\phi)$ with entries

$$(T_k(\phi))_{jk} = \phi_{j-k}, \qquad 0 \leqslant j, \quad k \leqslant k-1,$$

where ϕ_l are the Fourier coefficients of ϕ:

$$\phi_l = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}l\theta}\phi(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

Let us now state a result of Deift [1]. (See also earlier results of Widom [13, 14]).

Theorem 1. *Let $\varphi(t)$ be the symbol of the k-dimensional Toeplitz determinant $D_k[\varphi]$ that depends on a parameter t, then*

$$\frac{\mathrm{d}\log D_k[\varphi]}{\mathrm{d}t} = \int_{S^1}(1-\varphi)^{-1}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\sum_{j=1}^{2}F_j'(\zeta)G_j(\zeta)\,\mathrm{d}\zeta, \tag{3.1}$$

where $F_j(\zeta)$ and $G_j(\zeta)$ are the following:

$$\begin{aligned}(F_1, F_2)^T &= M_+(\zeta)(\zeta^k, 1)^T,\\ (G_1, G_2)^T &= \frac{1-\varphi}{2\pi i}M_+^{-T}(\zeta)(\zeta^{-k}, -1)^T.\end{aligned} \tag{3.2}$$

and the $'$ denotes differentiation with respect to ζ.

The matrix $M_+(\zeta)$ is the left-hand-side boundary value of the solution to the following Riemann–Hilbert problem

$$\begin{aligned}&(1)\quad M(\zeta) \text{ is analytic in } \zeta; \text{ on } \mathbb{C}/S^1,\\ &(2)\quad M_+(\zeta) = M_-(\zeta)\begin{pmatrix}\varphi & -(\varphi-1)\zeta^k\\ \zeta^{-k}(\varphi-1) & 2-\varphi\end{pmatrix}, \qquad \zeta\in S^1\\ &(3)\quad M(\zeta) = I + O(\zeta^{-1}), \qquad \zeta\to\infty,\end{aligned} \tag{3.3}$$

where S^1 is the unit circle that is oriented counter-clockwise.

The solution of the Riemann–Hilbert problem (3.3) and the Birkhoff factorization are related in the following way. Let H_0 and H_∞ be

$$\begin{aligned}H_0(\zeta) &= M(\zeta)\begin{pmatrix}\zeta^k & -1\\ 1 & 0\end{pmatrix},\\ H_\infty(\zeta) &= M(\zeta)\begin{pmatrix}1 & 0\\ \zeta^{-k} & 1\end{pmatrix}.\end{aligned} \tag{3.4}$$

Then H_0 and H_∞ solves the Birkhoff factorization (2.2) with $\phi = -\varphi$.

4. The tau function of the isomonodromic problem

The tau function of a general isomonodromic problem was introduced by Jimbo, Miwa and Ueno [2, 3]. The logarithmic derivative of the tau function has poles at the points $z, \tilde{z}, \tilde{w}, w$ in which the Birkhoff factorization problem is not solvable. It is an important object that is used to compute correlation functions in quantum field theory and is also related to Fredholm determinants and Grassmannian in the studies of integrable systems. In recent years, it also appears as local eigenvalue correlation functions in the study of random matrix theory [7–9]. In [5], the isomonodromic tau function was expressed in terms of the twistor data. In this section we shall use the result of [5] to establish a link between Toeplitz determinant and the tau function for the Ward ansätze.

In [6], it was shown that the Higgs fields in (1.4) are conjugate to

$$\begin{aligned} &\tilde{A}+B\sim-\frac{b}{2}\sigma_3, && A+\tilde{B}\sim-\frac{1}{2}(a-c+1)\,\sigma_3,\\ &\tilde{A}+\tilde{B}\sim\frac{1}{2}(b-c+1-k)\,\sigma_3, && A+B\sim\frac{1}{2}(a+k)\,\sigma_3, \end{aligned} \tag{4.1}$$

where $\sim$ means 'conjugate to' in the above.

By using the general theory of isomonodromic tau function in [5], we can express the tau function of the isomonodromic problem (1.4) in terms of H_0 and H_∞,

$$\frac{\mathrm{d}\log\tau}{\mathrm{d}t}=-\frac{b}{2}\mathrm{Res}_{\zeta=-t}\,\mathrm{Tr}\left(H_0^{-1}\frac{\mathrm{d}H_0}{\mathrm{d}\zeta}\frac{\sigma_3}{\zeta+t}\right), \tag{4.2}$$

where we have chosen $s=z=t$ and $w=\tilde{w}=\tilde{z}=1$ in (1.4).

Let us see how this derivative is related to the k-dimensional Toeplitz determinant with symbol $-\phi$.

5. Tau function and Toeplitz determinant

Let us first define Y_0 and Y_∞ to be the following functions in V_0 and V_∞:

$$Y_0=H_0\,(-\phi)^{\frac{\sigma_3}{2}}\zeta^{-\frac{k\sigma_3}{2}},\qquad Y_\infty=H_\infty(-\phi)^{\frac{\sigma_3}{2}}\zeta^{-\frac{k\sigma_3}{2}}, \tag{5.1}$$

then Y_0 and Y_∞ satisfy the following in $V_0\cap V_\infty$,

$$Y_0=Y_\infty\begin{pmatrix}\zeta^k & \zeta^k\\ 0 & \zeta^{-k}\end{pmatrix}. \tag{5.2}$$

By identifying $\varphi=-\phi$ in theorem 1 and using (3.4), we see that the Toeplitz determinant with symbol $-\phi$ satisfies the following:

$$\begin{aligned}\frac{\mathrm{d}\log D_k[-\phi]}{\mathrm{d}t}&=\frac{1}{2\pi\mathrm{i}}\oint_{S^1}\frac{\mathrm{d}\log\phi}{\mathrm{d}t}\,\mathrm{Tr}\left(Y_0'\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}Y_0^{-1}\right.\\ &\qquad\left.-\frac{1}{2}(\phi'\phi^{-2}+k\zeta^{-1})\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\mathrm{d}\zeta\\ &=\frac{1}{2\pi\mathrm{i}}\oint_{S^1}\frac{\mathrm{d}\log\phi}{\mathrm{d}t}\,\mathrm{Tr}\left(Y_0'\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}Y_0^{-1}\right).\end{aligned} \tag{5.3}$$

The following observation was due to A Its. First let us rewrite the above expression as

$$\begin{aligned}\frac{\mathrm{d}\log D_k[-\phi]}{\mathrm{d}t}&=\frac{1}{4\pi\mathrm{i}}\oint_{S^1}\frac{\mathrm{d}\log\phi}{\mathrm{d}t}\,\mathrm{Tr}\big(Y_0^{-1}Y_0'\sigma_3\big)\\ &\quad+\frac{1}{2\pi\mathrm{i}}\oint_{S^1}\frac{\mathrm{d}\log\phi}{\mathrm{d}t}\,\mathrm{Tr}\left(Y_0^{-1}Y_0'\begin{pmatrix}-\frac{1}{2} & -1\\ 0 & \frac{1}{2}\end{pmatrix}\right).\end{aligned} \tag{5.4}$$

Note that since

$$\begin{pmatrix}\zeta^k & \zeta^k\\ 0 & \zeta^{-k}\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & -1\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\zeta^k & \zeta^k\\ 0 & \zeta^{-k}\end{pmatrix}^{-1}=-\frac{\sigma_3}{2}, \tag{5.5}$$

by using (5.2) and (5.5) in (5.4), we obtain

$$\frac{\mathrm{d}\log D_k[-\phi]}{\mathrm{d}t} = \frac{1}{4\pi \mathrm{i}}\oint_{S^1} \frac{\mathrm{d}\log\phi}{\mathrm{d}t}\,\mathrm{Tr}\left(Y_0^{-1}Y_0'\sigma_3\right)\mathrm{d}\zeta - \frac{1}{4\pi \mathrm{i}}\oint_{S^1} \frac{\mathrm{d}\log\phi}{\mathrm{d}t}\left(\mathrm{Tr}\left(Y_\infty^{-1}Y_\infty'\sigma_3\right) + \frac{2k}{\zeta}\right)\mathrm{d}\zeta. \tag{5.6}$$

By using (5.1), we see that

$$\frac{\mathrm{d}\log D_k[-\phi]}{\mathrm{d}t} = \frac{1}{4\pi \mathrm{i}}\oint_{S^1} \frac{\mathrm{d}\log\phi}{\mathrm{d}t}\,\mathrm{Tr}\left(H_0^{-1}H_0'\sigma_3\right)\mathrm{d}\zeta - \frac{1}{4\pi \mathrm{i}}\oint_{S^1} \frac{\mathrm{d}\log\phi}{\mathrm{d}t}\left(\mathrm{Tr}\left(H_\infty^{-1}H_\infty'\sigma_3\right) + \frac{2k}{\zeta}\right)\mathrm{d}\zeta. \tag{5.7}$$

Since $\tilde{w} = 1$, $z = s = t$ and ϕ is a function of λ, μ and ζ only, we have,

$$\partial_\mu\phi = \partial_t\phi = -\frac{b}{\zeta+t}\phi. \tag{5.8}$$

If $|t| > 1$, then we can deform the unit circle S^1 in the first term of (5.7) into a close loop around $\zeta = -t$ and the unit circle in the second term into a close loop around $\zeta = \infty$ to obtain

$$\frac{\mathrm{d}D_k[-\phi]}{\mathrm{d}t} = -\frac{b}{2}\mathrm{Res}_{\zeta=-t}\,\mathrm{Tr}\left(H_0^{-1}H_0'\frac{\sigma_3}{\zeta+t}\right) - \frac{b}{2}\mathrm{Res}_{\zeta=\infty}\,\mathrm{Tr}\left(H_\infty^{-1}H_\infty'\frac{\sigma_3}{\zeta+t}\right). \tag{5.9}$$

Since $\frac{\mathrm{d}H_\infty}{\mathrm{d}\zeta} = O(\zeta^{-1})$ at $\zeta = \infty$, the above equation becomes

$$\frac{\mathrm{d}\log D_k[-\phi]}{\mathrm{d}t} = -\frac{b}{2}\mathrm{Res}_{\zeta=-t}\,\mathrm{Tr}\left(H_0^{-1}H_0'\frac{\sigma_3}{\zeta+t}\right) = \frac{\mathrm{d}\log\tau}{\mathrm{d}t}. \tag{5.10}$$

On the other hand, if $|t| < 1$, then the first integral in (5.7) vanishes and we obtain

$$\frac{\mathrm{d}\log D_k[-\phi]}{\mathrm{d}t} = -\frac{b}{2}\mathrm{Res}_{\zeta=-t}\,\mathrm{Tr}\left(H_\infty^{-1}H_\infty'\frac{\sigma_3}{\zeta+t}\right) - \frac{k}{t} = \frac{\mathrm{d}\log\tau}{\mathrm{d}t} - \frac{k}{t}. \tag{5.11}$$

>We have therefore shown the following theorem.

Theorem 2. *The tau function of the hypergeometric solutions of the sixth Painlevé equation obtained from the Ward ansätze (2.1) is given by*

$$\log\tau = \begin{cases} C_1 t^{-k} D_k[-\phi], & |t| < 1; \\ C_2 D_k[-\phi], & |t| > 1, \end{cases} \tag{5.12}$$

where C_1 and C_2 are some constants and $D_k[-\phi]$ is the k-dimensional Toeplitz determinant with the symbol $-\phi$.

Since the solution of the Painlevé VI equation is determined by its tau function, the above theorem classifies the hypergeometric solutions obtained in [6].

6. Concluding remark

We have classified the hypergeometric solutions of Painlevé VI obtained in [6] in terms of Toeplitz determinant. The method we used does not restrict to these solutions and can be applied to any solution constructed through the Ward ansätze. In particular, since all the six Painlevé equations can be obtained as reductions of the ASDYM equation, one can construct solutions to other Painlevé equations through the use of the Ward ansätze and theorem 2 will remain true for all these solutions.

It is worth mentioning that while the Ward construction solves the Birkhoff factorization problem (2.2) through inversion of a k-dimensional Hankel matrix (the matrix M in theorem 8.2.2 of [10]), we have represented the solutions through the k-dimensional Toeplitz determinant with symbol $-\phi$. This provides an alternative way of constructing solutions to the Ward ansätze via inversion of Toeplitz matrix and may provide new insights into the understanding of the relation between the Ward construction and the inversion of operators.

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